

The Yang Mills system and cyclic covering of abelian varieties

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Abstract

In this paper, we consider a dynamical system related to the Yang-Mills system for a field with gauge group $SU(2)$. We solve this system in terms of genus two hyperelliptic functions and we show that it is algebraic completely integrable in the generalized sense.

Key words. Integrable systems , Riemann surfaces, Abelian varieties, Surfaces of general type.

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1 Introduction

The problem of finding and integrating hamiltonian systems, has attracted a considerable amount of attention in recent years. Beside the fact that many integrable hamiltonian systems have been on the subject of powerful and beautiful theories of mathematics, another motivation for its study is: the concepts of integrability have been applied to an increasing number of physical systems, biological phenomena, population dynamics, chemical rate equations, to mention only a few. However, it seems still hopeless to describe or even to recognize with any facility, those hamiltonian systems which are integrable, though they are quite exceptional.

The resolution of the well known Korteweg-de-Vries (K-dV) equation has generated an enormous number of new ideas in the area of hamiltonian completely integrable systems. It has led to unexpected connections between mechanics, spectral theory, Lie algebra theory, algebraic geometry and even differential geometry. All these connections have generated renewed interest in the questions around complete integrability of finite and infinite dimensional systems, ordinary and partial differential equations. However given a hamiltonian system, it remains often hard to fit it into any of those general frameworks. But luckily, most of the problems possess the much richer

structure of the so called algebraic complete integrability (concept introduced et systematized by Adler and van Moerbeke). A dynamical system is algebraic completely integrable in the sense of Adler-van Moerbeke [1] if it can be linearized on a complex algebraic torus $\mathbb{C}^n/lattice$ (=abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equals to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines. Some results concerning geodesic flow on $SO(4)$ [1,8], Kowalewski's top [11], Hénon-Heiles system [14],...was obtained. However, besides the fact that many hamiltonian completely integrable systems posses this structure, another motivation for its study which sounds more modern is: algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate. Therefore there are hidden symmetries which have a group theoretical foundation. The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results. In fact, the overwhelming majority of dynamical systems, hamiltonian or not, are non-integrable and possess regimes of chaotic behavior in phase space.

In the present paper, we discuss an interesting interaction between complex geometry and dynamical systems. We shall be concerned with an integrable system which appears as covering of another algebraic completely integrable system. The invariant variety is covering of abelian variety and this system is algebraic completely integrable in the generalized sense.

2 The nonlinear Yang Mills equations for a field with gauge group $SU(2)$

We consider the Yang-Mills system for a field with gauge group $SU(2)$:

$$D_j F_{jk} = \partial_j F_{jk} + [A_j, F_{jk}] = 0,$$

where $F_{jk}, A_j \in T_e SU(2), 1 \leq j, k \leq 4$ and

$$F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k].$$

The self-dual Yang-Mills (SDYM) equations is a universal system for which some reductions include all classical tops from Euler to Kowalewski (0+1-dimensions), K-dV, Nonlinear Schrödinger, Sine-Gordon, Toda lattice and N-waves equations (1+1-dimensions), KP and D-S equations (2+1-dimensions),

etc... In the case of homogeneous double-component field,

$$\begin{aligned}\partial_j A_k &= 0, \quad (j \neq 1), \\ A_1 &= A_2 = 0, \\ A_3 &= n_1 U_1 \in su(2), \\ A_4 &= n_2 U_2 \in su(2),\end{aligned}$$

where n_i are $su(2)$ -generators, i.e., they satisfy commutation relations :

$$\begin{aligned}n_1 &= [n_2, [n_1, n_2]], \\ n_2 &= [n_1, [n_2, n_1]].\end{aligned}$$

The system becomes

$$\begin{aligned}\partial^2 U_1 + U_1 U_2^2 &= 0, \\ \partial^2 U_2 + U_2 U_1^2 &= 0.\end{aligned}$$

By setting

$$\begin{aligned}U_j &= q_j, \\ \frac{\partial U_j}{\partial t} &= p_j, \quad j = 1, 2,\end{aligned}$$

Yang-Mills equations are reduced to hamiltonian system with the hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 q_2^2).$$

The symplectic transformation

$$\begin{aligned}p_1 &\longleftarrow \frac{\sqrt{2}}{2}(p_1 + p_2), \\ p_2 &\longleftarrow \frac{\sqrt{2}}{2}(p_1 - p_2), \\ q_1 &\longleftarrow \frac{1}{2}(\sqrt[4]{2})(q_1 + iq_2), \\ q_2 &\longleftarrow \frac{1}{2}(\sqrt[4]{2})(q_1 - iq_2),\end{aligned}$$

takes this hamiltonian into

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}q_1^4 + \frac{1}{4}q_2^4 + \frac{1}{2}q_1^2 q_2^2. \quad (1)$$

We start with the hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + a_1 q_1^2 + a_2 q_2^2) + \frac{1}{4}q_1^4 + \frac{1}{4}a_3 q_2^4 + \frac{1}{2}a_4 q_1^2 q_2^2. \quad (2)$$

Note that if $a_1 = a_2 = 0$ and $a_3 = a_4 = 1$, we obtain the hamiltonian (1). It has been shown [10] that if $a_2 = 4a_1 \equiv 4a, a_3 = 16, a_4 = 6$, i.e.,

$$H_1 \equiv H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{a}{2}(q_1^2 + 4q_2^2) + \frac{1}{4}q_1^4 + 4q_2^4 + 3q_1^2q_2^2, \quad (3)$$

the corresponding system, i.e.,

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= -(a + q_1^2 + 6q_2^2)q_1, \\ \dot{p}_2 &= -2(2a + 3q_1^2 + 8q_2^2)q_2, \end{aligned} \quad (4)$$

is integrable, the second integral is

$$H_2 = aq_1^2q_2 + q_1^4q_2 + 2q_1^2q_2^3 - q_2p_1^2 + q_1p_1p_2, \quad (5)$$

but no description of solutions is given. We solve the system (4) in terms of genus two hyperelliptic functions. When one examines all possible singularities of the system (4), one finds that it possible for the variable q_1 to contain square root terms of the type $t^{1/2}$, which are strictly not allowed by the so called Painlevé test (i.e. the general solutions should have no movable singularities other than poles in the complex plane [5]).

Let $z \equiv (q_1, q_2, p_1, p_2) \in \mathbb{C}^4, t \in \mathbb{C}$ and $\Delta \subset \mathbb{C}^4$ a non-empty Zariski open set. By the functional independence of the integrals H_1, H_2 , the map

$$\varphi : (H_1, H_2) : \mathbb{C}^4 \longrightarrow \mathbb{C}^2,$$

is submersive, i.e., $dH_1(z), dH_2(z)$ are linearly independent on Δ . Let

$$\begin{aligned} \Omega &= \varphi(\mathbb{C}^4 \setminus \Delta), \\ &= \{b \equiv (b_1, b_2) \in \mathbb{C}^2 : \exists z \in \varphi^{-1}(b) \text{ with} \\ &\quad dH_1(z), dH_2(z) \text{ linearly dependent}\}, \end{aligned}$$

be the set of critical values of φ . We denote by $\overline{\Omega}$ the Zariski closure of Ω in \mathbb{C}^2 . The set $\{z \in \mathbb{C}^4 : \varphi(z) \in \mathbb{C}^2 \setminus \overline{\Omega}\}$ is a non-empty Zariski open set in \mathbb{C}^4 . Hence this set is everywhere dense in \mathbb{C}^4 for the usual topology. Let A be the complex affine variety defined by

$$\begin{aligned} A &= \varphi^{-1}(b), \\ &= \bigcap_{k=1}^2 \{z \in \mathbb{C}^4 : H_k(z) = b_k\}. \end{aligned} \quad (6)$$

For every $b \equiv (b_1, b_2) \in \mathbb{C}^2 \setminus \overline{\Omega}$, the fibre A is a smooth affine surface.

3 Laurent series solutions and algebraic curves

We show that the system (4) admits Laurent solutions in $t^{1/2}$, depending on three free parameters: u, v and w . These pole solutions restricted to the surface $A(6)$ are parameterized by two smooth curves $\mathcal{C}_{\varepsilon=\pm i}(8)$ of genus 4.

Recall that a system $\dot{z} = f(z)$ is weight-homogeneous with a weight ν_k going with each variable z_k if

$$f_k(\lambda^{\nu_i} z_1, \dots, \lambda^{\nu_n} z_n) = \lambda^{\nu_k+1} f'_k(z_1, \dots, z_n),$$

for all $\lambda \in \mathbb{C}$. The system (4) is weight-homogeneous with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that H_1 and H_2 have weight 4 and 5 respectively.

Theorem 1 *The system (4) admits Laurent solutions in $t^{1/2}$, depending on 3 free parameters: u, v and w . These solutions restricted to the surface $A(6)$ are parameterized by two smooth curves $\mathcal{C}_{\varepsilon=\pm i}(8)$ of genus 4.*

Proof. The system (4) possesses 3-dimensional family of Laurent solutions (principal balances) depending on three free parameters u, v and w . There are precisely two such families, labeled by $\varepsilon = \pm i$, and they are explicitly given as follows

$$\begin{aligned} q_1 &= \frac{1}{\sqrt{t}}(u - \frac{1}{2}u^3t + vt^2 + u^2(-\frac{11}{16}u^5 + \frac{1}{3}au + v)t^3 \\ &\quad + \frac{u}{4}(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w)t^4 + \dots), \\ q_2 &= \frac{\varepsilon\sqrt{2}}{4t}(1 + u^2t + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{8}u(24v - u^5)t^3 - 2\varepsilon\sqrt{2}wt^4 + \dots), \\ p_1 &= \frac{1}{t\sqrt{t}}(-\frac{1}{2}u - \frac{1}{4}u^3t + \frac{3}{2}vt^2 + \frac{5}{2}u^2(-\frac{11}{16}u^5 + \frac{1}{3}au + v)t^3 \\ &\quad + \frac{7u}{8}(\frac{41}{32}u^8 - au^4 + \frac{3}{2}u^3v + \frac{1}{6}a^2 - \frac{3\varepsilon\sqrt{2}}{2}w)t^4 + \dots), \\ p_2 &= \frac{\varepsilon\sqrt{2}}{4t^2}(-1 + \frac{1}{3}(2a - 3u^4)t^2 + \frac{1}{4}u(24v - u^5)t^3 - 6\varepsilon\sqrt{2}wt^4 + \dots). \end{aligned} \quad (7)$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter w linearly, leading to an equation connecting the two remaining parameters u and v :

$$2v^2 + \frac{1}{6}(15u^4 - 8a)uv - \frac{39}{32}u^{10} + \frac{7}{6}au^6 + \frac{2}{9}(a^2 + 9b_1)u^2 - \varepsilon\sqrt{2}b_2 = 0. \quad (8)$$

This defines two smooth curves \mathcal{C}_ε ($\varepsilon = \pm i$). Let $g(\mathcal{C}_\varepsilon)$ =genus of \mathcal{C}_ε , n = number of sheets and v = number of branch points. Then by the Riemann-Hurwitz's formula [17],

$$g(\mathcal{C}_\varepsilon) = -n + 1 + \frac{v}{2} = -2 + 1 + \frac{10}{2} = 4,$$

which finishes the proof of the theorem.

4 Linearizing the flow in terms of genus two hyperelliptic functions

Theorem 2 *The system of differential equations (4) can be integrated in terms of genus 2 hyperelliptic functions.*

Proof. We set

$$\begin{aligned} q_2 &= s_1 + s_2, \\ q_1^2 &= -4s_1s_2, \\ p_2 &= \dot{s}_1 + \dot{s}_2, \\ q_1p_1 &= -2(\dot{s}_1s_2 + s_1\dot{s}_2). \end{aligned}$$

The latter equation together with the second implies that

$$p_1^2 = -\frac{(\dot{s}_1s_2 + s_1\dot{s}_2)^2}{s_1s_2}.$$

In term of these new variables, equations (3) and (5) take the following form

$$\begin{aligned} &(s_1 - s_2) (s_2(\dot{s}_1)^2 - s_1(\dot{s}_2)^2) \\ &+ 4s_1s_2 (2s_1^4 + 2s_1^3s_2 + 2s_1^2s_2^2 + 2s_1s_2^3 + 2s_2^4 + as_1^2 + as_1s_2 + as_2^2) \\ &- 2b_1s_1s_2 = 0, \\ &(s_1 - s_2) (s_2^2(\dot{s}_1)^2 - s_1^2(\dot{s}_2)^2) \\ &+ 4s_1^2s_2^2 (s_1 + s_2) (a + 2s_1^2 + 2s_2^2) + b_2s_1s_2 = 0. \end{aligned}$$

These equations are solved linearly for $(\dot{s}_1)^2$ and $(\dot{s}_2)^2$ as

$$\begin{aligned} (\dot{s}_1)^2 &= \frac{s_1(-8s_1^5 - 4as_1^3 + 2b_1s_1 + b_2)}{(s_1 - s_2)^2}, \\ (\dot{s}_2)^2 &= \frac{s_2(-8s_2^5 - 4as_2^3 + 2b_1s_2 + b_2)}{(s_1 - s_2)^2}, \end{aligned}$$

which leads immediately to the following equations for s_1 and s_2 :

$$\begin{aligned} \dot{s}_1 &= \frac{ds_1}{dt} = \frac{\sqrt{P_6(s_1)}}{s_1 - s_2}, \\ \dot{s}_2 &= \frac{ds_2}{dt} = \frac{\sqrt{P_6(s_2)}}{s_2 - s_1}, \end{aligned}$$

where $P_6(s)$ is a polynomial of degree 6 of the form

$$P_6(s) = s(-8s^5 - 4as^3 + 2b_1s + b_2).$$

These equations can be integrated by the abelian mapping

$$\mathcal{H} \longrightarrow \text{Jac}(\mathcal{H}) = \mathbb{C}^2/\Lambda, \quad (p_1, p_2) \longmapsto (\xi_1, \xi_2),$$

where the hyperelliptic curve \mathcal{H} of genus 2 is given by the equation

$$\zeta^2 = P_6(s),$$

Λ is the lattice generated by the vectors $n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2$, Ω is the matrix of period of the curve \mathcal{H} , $p_1 = (s_1, \sqrt{P_6(s_1)})$, $p_2 = (s_2, \sqrt{P_6(s_2)})$,

$$\begin{aligned} \xi_1 &= \int_{p_0}^{p_1} \omega_1 + \int_{p_0}^{p_2} \omega_1, \\ \xi_2 &= \int_{p_0}^{p_1} \omega_2 + \int_{p_0}^{p_2} \omega_2, \end{aligned}$$

where p_0 is a fixed point and (ω_1, ω_2) is a canonical basis of holomorphic differentials on \mathcal{H} , i.e.,

$$\begin{aligned} \omega_1 &= \frac{ds}{\sqrt{P_6(s)}}, \\ \omega_2 &= \frac{s ds}{\sqrt{P_6(s)}}. \end{aligned}$$

We have

$$\begin{aligned} \frac{ds_1}{\sqrt{P_6(s_1)}} - \frac{ds_2}{\sqrt{P_6(s_2)}} &= 0, \\ \frac{s_1 ds_1}{\sqrt{P_6(s_1)}} - \frac{s_2 ds_2}{\sqrt{P_6(s_2)}} &= dt, \end{aligned}$$

and hence the problem can be integrated in terms of genus 2 hyperelliptic functions of time. This ends the proof of the theorem.

5 A five-dimensional system

We have seen that it possible for the variables q_1 and p_1 to contain square root terms of the type \sqrt{t} , which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing some new variables z_1, \dots, z_5 , which restores the Painlevé property to the system. Indeed, let

$$\varphi : A \longrightarrow \mathbb{C}^5, \quad (q_1, q_2, p_1, p_2) \longmapsto (z_1, z_2, z_3, z_4, z_5), \quad (9)$$

be a morphism on the affine variety $A(6)$ where z_1, \dots, z_5 are defined as

$$z_1 = q_1^2, \quad z_2 = q_2, \quad z_3 = p_2, \quad z_4 = q_1 p_1, \quad z_5 = 2q_1^2 q_2^2 + p_1^2.$$

The morphism (9) maps the vector field (4) into the system

$$\begin{aligned}
\dot{z}_1 &= 2z_4, \\
\dot{z}_2 &= z_3, \\
\dot{z}_3 &= -4az_2 - 6z_1z_2 - 16z_2^3, \\
\dot{z}_4 &= -az_1 - z_1^2 - 8z_1z_2^2 + z_5, \\
\dot{z}_5 &= -8z_2^2z_4 - 2az_4 - 2z_1z_4 + 4z_1z_2z_3,
\end{aligned} \tag{10}$$

in five unknowns having three quartic invariants

$$\begin{aligned}
F_1 &= \frac{1}{2}z_5 + 2z_1z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}az_1 + 2az_2^2 + \frac{1}{4}z_1^2 + 4z_2^4, \\
F_2 &= az_1z_2 + z_1^2z_2 + 4z_1z_2^3 - z_2z_5 + z_3z_4, \\
F_3 &= z_1z_5 - 2z_1^2z_2^2 - z_4^2.
\end{aligned} \tag{11}$$

This system is completely integrable and the hamiltonian structure is defined by the Poisson bracket

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J \frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^5 J_{kl} \frac{\partial F}{\partial z_k} \frac{\partial H}{\partial z_l},$$

where

$$\frac{\partial H}{\partial z} = \left(\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3}, \frac{\partial H}{\partial z_4}, \frac{\partial H}{\partial z_5} \right)^\top,$$

and

$$J = \begin{bmatrix} 0 & 0 & 0 & 2z_1 & 4z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4z_1z_2 \\ -2z_1 & 0 & 0 & 0 & 2z_5 - 8z_1z_2^2 \\ -4z_4 & 0 & 4z_1z_2 & -2z_5 + 8z_1z_2^2 & 0 \end{bmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The system (10) can be written as

$$\dot{z} = J \frac{\partial H}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

where $H = F_1$. The second flow commuting with the first is regulated by the equations

$$\dot{z} = J \frac{\partial F_2}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

and is written explicitly as

$$\begin{aligned}
\dot{z}_1 &= 2z_1z_3 - 4z_2z_4, \\
\dot{z}_2 &= z_4, \\
\dot{z}_3 &= z_5 - 8z_1z_2^2 - az_1 - z_1^2, \\
\dot{z}_4 &= -2az_1z_2 - 4z_1^2z_2 - 2z_2z_5, \\
\dot{z}_5 &= -4az_2z_4 - 4z_1z_2z_4 - 16z_2^3z_4 - 2z_3z_5 + 8z_1z_2^2z_3.
\end{aligned}$$

These vector fields are in involution, i.e.,

$$\{F_1, F_2\} = \left\langle \frac{\partial F_1}{\partial z}, J \frac{\partial F_2}{\partial z} \right\rangle = 0,$$

and the remaining one is Casimir, i.e.,

$$J \frac{\partial F_3}{\partial z} = 0.$$

Let B be the complex affine variety defined by

$$B = \bigcap_{k=1}^2 \{z : F_k(z) = c_k\} \subset \mathbb{C}^5, \quad (12)$$

for generic $(c_1, c_2, c_3) \in \mathbb{C}^3$. We have shown in [18], that

- a) The the system (10) can be integrated in genus 2 hyperelliptic functions.
- b) The system (10) possesses Laurent series solutions which depend on 4 free parameters : α, β, γ and θ :

$$\begin{aligned} z_1 &= \frac{1}{t}(\alpha - \alpha^2 t + \beta t^2 + \frac{1}{6}\alpha(3\beta - 9\alpha^3 + 4a\alpha)t^3 + \gamma t^4 + \dots), \\ z_2 &= \frac{\varepsilon\sqrt{2}}{4t}(1 + \alpha t + \frac{1}{3}(-3\alpha^2 + 2a)t^2 + \frac{1}{2}(3\beta - \alpha^3)t^3 - 2\varepsilon\sqrt{2}\theta t^4 + \dots), \\ z_3 &= \frac{\varepsilon\sqrt{2}}{4t^2}(-1 + \frac{1}{3}(-3\alpha^2 + 2a)t^2 + (3\beta - \alpha^3)t^3 - 6\varepsilon\sqrt{2}\theta t^4 + \dots), \quad (13) \\ z_4 &= \frac{1}{2t^2}(-\alpha + \beta t^2 + \frac{1}{3}\alpha(3\beta - 9\alpha^3 + 4a\alpha)t^3 + 3\gamma t^4 + \dots), \\ z_5 &= \frac{1}{t}(-\frac{1}{3}a\alpha + \alpha^3 - \beta + (3\alpha^4 - a\alpha^2 - 3\alpha\beta)t \\ &\quad + (4\varepsilon\sqrt{2}\alpha\theta + 2\gamma + \frac{8}{3}a\alpha^3 - \frac{1}{3}a\beta - \alpha^2\beta - 3\alpha^5 - \frac{4}{9}a^2\alpha)t^2 + \dots), \end{aligned}$$

with $\varepsilon = \pm i$. These meromorphic solutions restricted to the surface $B(12)$ are parameterized by two isomorphic smooth hyperelliptic curves $\mathcal{H}_{\varepsilon=\pm i}$ of genus 2 :

$$\beta^2 + \frac{2}{3}(3\alpha^2 - 2a)\alpha\beta - 3\alpha^6 + \frac{8}{3}a\alpha^4 + \frac{4}{9}(a^2 + 9c_1)\alpha^2 - 2\varepsilon\sqrt{2}c_2\alpha + c_3 = 0, \quad (14)$$

- c) The variety $B(12)$ is embedded in \mathbb{P}^{15} and generically is the affine part of an abelian surface \tilde{B} , more precisely the jacobian of a genus 2 curve. The reduced divisor at infinity

$$\tilde{B} \setminus B = \mathcal{H}_i + \mathcal{H}_{-i},$$

consists of two smooth isomorphic genus 2 curves $\mathcal{H}_\varepsilon(14)$, that intersect in only one point at which they are tangent to each other. The system of

differential equations (10) is algebraically completely integrable and the corresponding flows evolve on \tilde{B} .

Observe that the reflection σ on the affine variety B amounts to the flip

$$\sigma : (z_1, z_2, z_3, z_4, z_5) \longmapsto (z_1, z_2, -z_3, -z_4, z_5),$$

changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of \mathbb{C}^2 to the time flip $(t_1, t_2) \mapsto (-t_1, -t_2)$ on \tilde{B} , where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2} . The involution σ acts on the parameters of the Laurent solution (13) as follows

$$\sigma : (t, \alpha, \beta, \gamma, \theta, \varepsilon) \longmapsto (-t, -\alpha, -\beta, -\gamma, -\theta, -\varepsilon),$$

interchanges the curves $\mathcal{H}_{\varepsilon=\pm i}$ (14). Geometrically, this involution interchanges \mathcal{H}_i and \mathcal{H}_{-i} , i.e., $\mathcal{H}_{-i} = \sigma\mathcal{H}_i$.

The asymptotic solution (7) can be read off from (13) and the change of variable :

$$q_1 = \sqrt{z_1}, \quad q_2 = z_2, \quad p_1 = z_4/q_1, \quad p_2 = z_3.$$

The function z_1 has a simple pole along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ and a double zero along a hyperelliptic curve of genus 2 defining a double cover of \tilde{B} ramified along $\mathcal{H}_i + \mathcal{H}_{-i}$.

6 Generalized algebraic completely integrable system

Applying the method explained in Piovan [22], we show that the invariant variety $A(6)$ can be completed as a cyclic double cover \overline{A} of the jacobian of a genus curve, ramified along a divisor $\mathcal{H}_i + \mathcal{H}_{-i}$ where \mathcal{H}_i and \mathcal{H}_{-i} are two isomorphic hyperelliptic curves (14) of genus 2 that intersect in only one point at which they are tangent to each other. Moreover, \overline{A} is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$ and the resolution \tilde{A} of \overline{A} is a surface of general type with invariants : Euler characteristic of $\tilde{A} = \chi(\tilde{A}) = 1$ and geometric genus of $\tilde{A} = p_g(\tilde{A}) = 2$. Consequently, the system (4) is algebraic completely integrable in the generalized sense.

Theorem 3 *The invariant surface $A(6)$ can be completed as a cyclic double cover \overline{A} of the abelian surface \tilde{B} (the jacobian of a genus 2 curve), ramified along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$. The system (4) is algebraic complete integrable in the generalized sense. Moreover, \overline{A} is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$ and the resolution \tilde{A} of \overline{A} is a surface of general type with invariants : $\chi(\tilde{A}) = 1$ and $p_g(\tilde{A}) = 2$.*

Proof. We have shown that the morphism φ (9) maps the vector field (4) into an algebraic completely integrable system (10) in five unknowns and the affine variety A (6) onto the affine part B (12) of an abelian variety \tilde{B} (more precisely the jacobian of a genus 2 curve with $\tilde{B} \setminus B = \mathcal{H}_i + \mathcal{H}_{-i}$). Observe that $\varphi : A \rightarrow B$ is an unramified cover. The curves \mathcal{C}_ε (8) play an important role in the construction of a compactification \overline{A} of A . Let us denote by G a cyclic group of two elements $\{-1, 1\}$ on

$$V_\varepsilon^j = U_\varepsilon^j \times \{\tau \in \mathbb{C} : 0 < |\tau| < \delta\},$$

where $\tau = t^{1/2}$ and U_ε^j is an affine chart of \mathcal{C}_ε for which the Laurent solutions (7) are defined. The action of G is defined by

$$(-1) \circ (u, v, \tau) = (-u, -v, -\tau),$$

and is without fixed points in V_ε^j . So we can identify the quotient V_ε^j/G with the image of the smooth map $h_\varepsilon^j : V_\varepsilon^j \rightarrow A$ defined by the expansions (7). We have

$$(-1, 1) \cdot (u, v, \tau) = (-u, -v, \tau),$$

and

$$(1, -1) \cdot (u, v, \tau) = (u, v, -\tau),$$

i.e., $G \times G$ acts separately on each coordinate. Thus, identifying V_ε^j/G^2 with the image of $\varphi \circ h_\varepsilon^j$ in B . Note that $A_\varepsilon^j = V_\varepsilon^j/G$ is smooth (except for a finite number of points) and the coherence of the A_ε^j follows from the coherence of V_ε^j and the action of G . Now by taking A and by gluing on various varieties $A_\varepsilon^j \setminus \{\text{some points}\}$, we obtain a smooth complex manifold \hat{A} which is a double cover of the abelian variety \tilde{B} ramified along $\mathcal{H}_i + \mathcal{H}_{-i}$, and therefore can be completed to an algebraic cyclic cover of \tilde{B} . To see what happens to the missing points, we must investigate the image of $\mathcal{C}_\varepsilon \times \{0\}$ in $\cup A_\varepsilon^j$. The quotient $\mathcal{C}_\varepsilon \times \{0\}/G$ is birationally equivalent to the smooth hyperelliptic curve Γ_ε of genus 2 :

$$2w^2 + \frac{1}{6}(15z^2 - 8a)zw + z(-\frac{39}{32}z^5 + \frac{7}{6}az^3 + \frac{2}{9}(a^2 + 9b_1)z - \varepsilon\sqrt{2}b_2) = 0,$$

where $w = uv, z = u^2$. The curve Γ_ε is birationally equivalent to \mathcal{H}_ε . The only points of \mathcal{C}_ε fixed under $(u, v) \mapsto (-u, -v)$ are the two points at ∞ , which correspond to the ramification points of the map

$$\mathcal{C}_\varepsilon \times \{0\} \xrightarrow{2-1} \Gamma_\varepsilon : (u, v) \mapsto (z, w),$$

and coincides with the points at ∞ of the curve \mathcal{H}_ε . Then the variety \hat{A} constructed above is birationally equivalent to the compactification \overline{A} of the generic invariant surface A . So \overline{A} is a cyclic double cover of the abelian surface \tilde{B} (the jacobian of a genus 2 curve) ramified along the divisor $\mathcal{H}_i + \mathcal{H}_{-i}$, where

\mathcal{H}_i and \mathcal{H}_{-i} intersect each other in a tacnode. It follows that The system (4) is algebraic complete integrable in the generalized sense. Moreover, \overline{A} is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{H}_i + \mathcal{H}_{-i}$. In term of an appropriate local holomorphic coordinate system (x, y, z) , the local analytic equation about this singularity is $x^4 + y^2 + z^2 = 0$. Now, let \tilde{A} be the resolution of singularities of \overline{A} , $\chi(\tilde{A})$ be the Euler characteristic of \tilde{A} and $p_g(\tilde{A})$ the geometric genus of \tilde{A} . Then \tilde{A} is a surface of general type with invariants : $\chi(\tilde{A}) = 1$ and $p_g(\tilde{A}) = 2$. This concludes the proof of the theorem.

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